Torelli type theorems for gravitational instantons

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Gravitational instanton

Definition

A gravitational instanton is a non-compact complete non-flat hyperkähler 4-manifold such that $\int_X |\mathrm{Rm}|^2 < \infty$.

Definition

A hyperkähler 4-manifold is a 4-dimensional Riemannian manifold (X, g) with three Kähler structures (X, g, I), (X, g, J), (X, g, K) such that $IJ = K$.

Gravitational instanton

Definition

A gravitational instanton is a non-compact complete non-flat hyperkähler 4-manifold such that $\int_X |\mathrm{Rm}|^2 < \infty$.

Some people use different definitions:

- If the manifold is compact, it is called a K3 surface.
- If the manifold is flat, then it must be $\mathbb{R} \times T^3$, $\mathbb{R}^2 \times T^2$, $\mathbb{R}^3 \times S^1$ or $\mathbb{R}^4$.
- If the manifold is simply connected, the metric is hyperkähler if and only it is Ricci-flat ("gravitational") and the curvature is anti-self-dual ("instanton").
- Some people use different curvature decay conditions.

Gravitational instanton

Question

- *Understand all possible asymptotic structures.*
- *Understand all hyperkähler metrics with a given asymptotic structure.*

Asymptotical geometry of gravitation instantons

Theorem (C.-Chen)

Under the faster than quadratic curvature decay condition $|Rm| = O(r^{-2-\epsilon})$, gravitational instantons must be ALE (Asymptotically Locally Euclidean), ALF (Asymptotically Locally Flat), ALG, or ALH ("G", "H" are the letters after "E" and "F".)

Theorem (Sun-Zhang)

Any gravitational instanton X satisfying $\int_X |\mathrm{Rm}|^2 < \infty$ (so that $|\mathrm{Rm}| = O(r^{-2})$ by Cheeger-Tian) but not $|Rm| = O(r^{-2-\epsilon})$ must be ALG or ALH*.*

Asymptotical geometry of gravitation instantons

	Curvature	Volume	Tangent cone at infinity
ALE	$O(r^{-6})$	$O(r^4)$	$\mathbb{R}^4/\Gamma$
ALF-A$_k$	$O(r^{-3})$	$O(r^3)$	$\mathbb{R}^3$
ALF-D$_k$	$O(r^{-3})$	$O(r^3)$	$\mathbb{R}^3/\mathbb{Z}_2$
ALG	$O(r^{-2-\delta}), \delta = \min_{n\in\mathbb{Z}, n<2\beta} \frac{2\beta-n}{\beta}$	$O(r^2)$	$\mathbb{C}_\beta$
ALG*	$O(r^{-2}(\log r)^{-1})$	$O(r^2)$	$\mathbb{R}^2/\mathbb{Z}_2$
ALH	$O(e^{-\delta r})$	$O(r)$	$[0,\infty)$
ALH*	$O(r^{-2})$	$O(r^{4/3})$	$[0,\infty)$

Asymptotical geometry of gravitation instantons

Definition

A gravitational instanton (X, g, I, J, K) is called ALE of order ϵ if there exist a bounded domain $X_R \subset X$, and a diffeomorphism

$$\Phi : (\mathbb{C}^2 \setminus B_R(0))/\Gamma \to X \setminus X_R$$

such that

$$|\Phi^* g - g_{\mathbb{C}^2}|_{g_{\mathbb{C}^2}} = O(r^{-\epsilon}), |\Phi^* I - I_{\mathbb{C}^2}|_{g_{\mathbb{C}^2}} = O(r^{-\epsilon}),$$

$$|\Phi^* J - J_{\mathbb{C}^2}|_{g_{\mathbb{C}^2}} = O(r^{-\epsilon}), |\Phi^* K - K_{\mathbb{C}^2}|_{g_{\mathbb{C}^2}} = O(r^{-\epsilon}).$$

Definition

We get ALF, ALG, ALH, ALG*, and ALH* examples if we use different standard models to replace $(\mathbb{C}^2 \setminus B_R(0))/\Gamma$. In this talk, we focus on ALG and ALG* examples.

ALG* Model metric

Example (ALG* model)

Let U be the set $(\mathbb{R}^2 \setminus B_R(0)) \times \mathbb{S}^1$ and $V = \kappa_0 + \frac{\nu}{\pi} \log r$, $\nu = 1, 2, 3, 4$, be a harmonic function on U. There exists an S^1 fibration E with degree 2ν on U such that the connection 1-form α satisfies

$$d\alpha = *_{g_U} dV.$$

then $\mathbb{Z}_2$ quotient of the metric

$$g_E = L^2(V g_U + V^{-1}\alpha^2)$$

on E is called the ALG^*_ν model or the ALG*-$D_{4-\nu}$ model.

ALG model

Definition (ALG model)

Suppose $\beta \in (0,1]$ and $\tau \in \mathbb{H} = \{\tau | \mathrm{Im}\tau > 0\}$ are parameters in the following table:

β	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{3}$	$\frac{2}{3}$
τ	$\in \mathbb{H}$	$e^{2\pi i/3}$	$e^{2\pi i/3}$	i	i	$e^{2\pi i/3}$	$e^{2\pi i/3}$

Suppose $l > 0$ is any scaling parameter. Let E be the manifold obtained by identifying (u, v) with $(e^{2\pi i\beta}u, e^{-2\pi i\beta}v)$ in the space

$$\{(u,v)|\mathrm{arg}u \in [0, 2\pi\beta], |u| \geq R\} \subset (\mathbb{C} - B_R) \times \mathbb{C}/(\mathbb{Z}l \oplus \mathbb{Z}\tau l).$$

Then there is a flat hyperkähler metric g_0 on E such that $\omega^1 = \frac{i}{2}(du \wedge d\bar{u} + dv \wedge d\bar{v})$ and $\omega^+ = \omega^2 + i\omega^3 = du \wedge dv$. It is called the ALG model.

Gravitational instanton

Question

- *Understand all possible asymptotic structures.*
- *Understand all hyperkähler metrics with a given asymptotic structure.*

K3 surfaces

Definition

A K3 surface is a compact non-flat hyperkähler 4-manifold.

Theorem (Kodaira)

Any K3 surface is diffeomorphic to the Kummer's surface $\widetilde{T^4/\mathbb{Z}_2}$.

They are called K3 surfaces in honor of Kummer, Kähler, Kodaira and the K2 mountain.

Torelli-type theorem for K3 surfaces

Theorem (Burns-Rapoport, Todorov, Looijenga-Peters, Siu, Anderson)

Let X be the smooth 4-manifold which underlies the minimal resolution of $\mathbb{T}^4/\mathbb{Z}_2$. Let Ω be the space of three cohomology classes $[\alpha^1], [\alpha^2], [\alpha^3] \in H^2(X, \mathbb{R})$ which satisfy the following conditions:

- *(Integrability)* $\int_X \alpha^i \wedge \alpha^j = 2\delta_{ij}V$.
- *(Non-degeneracy) For any* $[\Sigma] \in H_2(X, \mathbb{Z})$ *with* $[\Sigma]^2 = -2$, *there exists* $i \in \{1, 2, 3\}$ *with* $[\alpha^i][\Sigma] \neq 0$.

Ω *has two components* Ω^+ *and* Ω^-. *For any* $([\alpha^1], [\alpha^2], [\alpha^3]) \in \Omega^+$, *there exists on* X *a hyperkähler structure for which the cohomology classes of the Kähler forms* $[\omega^i]$ *are the given* $[\alpha^i]$. *It is unique up to tri-holomorphic isometries which induce identity on* $H_2(X, \mathbb{Z})$.

Torelli-type theorem for gravitational instantons

	Torelli	Volume	Tangent cone at infinity
ALE	Kronheimer	$O(r^4)$	$\mathbb{R}^4/\Gamma$
ALF-A_k	Minerbe	$O(r^3)$	$\mathbb{R}^3$
ALF-D_k	C.-Chen	$O(r^3)$	$\mathbb{R}^3/\mathbb{Z}_2$
ALG		$O(r^2)$	$\mathbb{C}_\beta$
ALG*		$O(r^2)$	$\mathbb{R}^2/\mathbb{Z}_2$
ALH	C.-Chen	$O(r)$	$[0, \infty)$
ALH*	Collins-Jacob-Lin (Uniqueness)	$O(r^{4/3})$	$[0, \infty)$

We discuss the ALG and ALG* cases in this talk.

Topology of gravitational instantons

Theorem (C.-Chen)

Any ALG gravitational instanton is diffeomorphic to a rational elliptic surface minus a singular fiber.

Theorem (C.-Viaclovsky)

Any ALG^* gravitational instanton is diffeomorphic to a rational elliptic surface minus a singular fiber.

Corollary (C.-Viaclovsky)

All ALG gravitational instantons with the same β are diffeomorphic to each other. All ALG^*_ν gravitational instantons with the same ν are diffeomorphic to each other. Moreover, we can fix the coordinates near infinity.

Rational elliptic surface

Definition (Elliptic surface)

A complex surface M is called an elliptic surface if there exists a holomoprhic map π from M to a Riemann surface B such that for all except finitely many points on B, the inverse image is an elliptic curve. They are called regular fibers. The inverse image of the finitely many points are called singular fibers.

Definition (Rational elliptic surface)

Let f, g be homogenous polynomials with degree 3 in three variables. Then f/g is a map from the blow up of $\mathbb{CP}^2$ at the 9 points $\{f = g = 0\}$ to $\mathbb{CP}^1$. It is called a rational elliptic surface.

Rational elliptic surface

Theorem (Kodaira)

Singular fibers on any elliptic surface can be classified.

Remark

The type of the singular fibers on rational elliptic surfaces must be I_0^*, II, II^*, III, III^*, IV, IV^* (finite monodromy fibers), I_ν, $\nu = 1, 2, ..., 9$, or I_ν^*, $\nu = 1, 2, ...4$ (infinite monodromy fibers).

Tian-Yau and Hein's construction

- Yau solved Calabi's conjeture on compact manifolds, it was the key part of the proof of the existence part of the K3 Torelli theorem.
- On non-compact manifolds, Tian-Yau did the same thing assuming good background metrics. Their background metric provides ALH^* gravitational instantons.
- Hein found more background metrics on a rational elliptic surface minus a singular fiber and found examples of ALG (finite monodromy), ALG^* (I_ν^*), ALH^* (I_ν) gravitational instantons using Tian-Yau's theorem.

Tian-Yau and Hein's construction

Theorem (Tian-Yau, Hein)

Let (S, I) be a rational elliptic surface with a type I_ν^* fiber D. For any $\kappa_0 \in \mathbb{R}$, any Kähler form ω on S, and any rational 2-form $\Omega = \omega_2 + i\omega_3$ on S with $div(\Omega) = -D$, there exist $c > 0$, $L > 0$, and a smooth function $\varphi : X \to \mathbb{R}$, where $X \equiv S \setminus D$, such that

$$(X, g, \omega_1 = \omega + i\partial\bar{\partial}\varphi, c \cdot \omega_2, c \cdot \omega_3)$$

is an ALG^* gravitational instanton with parameters ν, κ_0, and L, where g is the metric determined by ω_1 and the elliptic complex structure I.

Classfication of ALG* gravitational instantons

Theorem (C.-Viaclovsky)

Conversely, let $(X, g, \boldsymbol{\omega})$ *be an ALG** *gravitational instanton with parameters* ν, κ_0*, and* L*. Then* $\nu \leq 4$*, and* X *can be compactified to a rational elliptic surface* S *with global section by adding a Kodaira singular fiber* D *of type* I_ν^* *at infinity, with respect to the complex structure* I*. The* 2*-form* $\Omega = \omega_2 + i\omega_3$ *is a rational 2-form on* S *with div*$(\Omega) = -D$*. Furthermore, we can choose* S *so that there exist a Kähler form* ω *on* S*, and a smooth function* $\varphi : X \to \mathbb{R}$*, satisfying*

$$\omega_1 = \omega + i\partial\bar{\partial}\varphi.$$

This is similar to C.-Chen's theorem in the ALG case.

Counterexample of ALG Torelli

Theorem (C.-Chen)

When $\frac{1}{2} < \beta < 1$, the order of any ALG gravitational instanton can be improved to $2 - \frac{1}{\beta} \in (0, 1)$. Moreover, there exist distinct examples with the same $[\omega_i]$. When $\beta \leq \frac{1}{2}$, the order of any ALG gravitational instanton can be improved to 2.

Theorem (C.-Viaclovsky)

When $\frac{1}{2} < \beta < 1$, each ALG gravitational instanton of order 2 corresponds to a two-parameter family of ALG gravitational instantons of order $2 - \frac{1}{\beta} \in (0, 1)$ with the same $[\omega_i]$.

Theorem (C.-Viaclovsky-Zhang)

The order of any ALG* gravitational instanton can be improved to 2.

Period map

For any ALG or ALG* gravitational instanton $(X, \omega_1^0, \omega_2^0, \omega_3^0)$ of order 2 and another ALG or ALG* gravitational instanton $(X, \omega_1, \omega_2, \omega_3)$ of order 2 with the same coordinates at infinity. Then the period map is defined by

$$(\omega_1 - \omega_1^0, \omega_2 - \omega_2^0, \omega_3 - \omega_3^0) \in \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H},$$

where

$$\mathcal{H} = \mathrm{Ima}(H^2_{cpt}(X) \to H^2(X)) = \{[\omega] \in H^2(X), \int_D \omega = 0\}.$$

Torelli-type theorem for gravitational instantons

Theorem (C.-Viaclovsky-Zhang)

The period map is injective up to a diffeomorphism which fixes $H^2(X)$.

Theorem (C.-Viaclovsky-Zhang)

The image of the period map is open in $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$.

Conjecture

The image of the period map is $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$ *if we allow orbifolds.*

Uniqueness parts of Torelli-type theorems

Theorem (C.-Viaclovsky-Zhang)

The period map is injective up to a diffeomorphism which fixes g, I, J, K, and $H^2(X)$.

The key idea is to use them as building blocks of a gluing construction to obtain a hyperKähler metric on the K3 surface and then use the K3 Torelli theorem. Actually, the same idea was used by C.-Chen to prove the uniqueness part of the ALH Torelli theorem.

Elliptic K3 surface

Definition (Elliptic surface)

A complex surface M is called an elliptic surface if there exists a holomoprhic map π from M to a Riemann surface B such that for all except finitely many points on B, the inverse image is an elliptic curve. They are called regular fibers. The inverse image of the finitely many points are called singular fibers.

Theorem (Kodaira)

Singular fibers on any elliptic surface can be classified.

Remark

If the elliptic surface is a K3 surface, then B must be $\mathbb{CP}^1$ and the type of the singular fibers must be I_0^*, II, II^*, III, III^*, IV, IV^* (finite monodromy fibers), I_ν, or I_ν^* $(\nu = 1, 2, \ldots)$ (infinite monodromy fibers).

Elliptic K3 surface

As the first step, we fix the elliptic complex structure and study collapsing hyperkähler metrics. This was done by C.-Viaclovsky-Zhang in 2020.
Then in 2021, we allow the change of complex structure in order to prove the Torelli-type theorems for ALG and ALG* gravitational instantons.

Gluing of ALG and ALG* gravitational instantons

Theorem (C.-Viaclovsky-Zhang)

For any elliptic K3 surface, we can glue the Greene-Shapere-Vafa-Yau's semi-flat metric with isotrivial order 2 ALG gravitational instantons near finite monodromy fibers, multi-Ooguri-Vafa metrics near I_ν fibers, and the $\mathbb{Z}_2$ quotients of the multi-Ooguri-Vafa metrics together with Eguchi-Hanson metrics near I_ν^* fibers to get a hyperKähler metric on the K3 surface without changing the complex structure.

Remark

Before our work, Gross-Wilson studied the case when the elliptic K3 surface has 24 I_1 fibers. This is the generic case.

Isotrivial ALG spaces

Definition (Isotrivial ALG spaces)

Consider the following $\beta \in (0,1]$ and $\tau \in \mathbb{H} = \{\tau | \mathrm{Im}\tau > 0\}$:

D_0	I_0^*	II^*	II	III^*	III	IV^*	IV
D_∞	I_0^*	II	II^*	III	III^*	IV	IV^*
β	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{3}$	$\frac{2}{3}$
τ	$\in \mathbb{H}$	$e^{2\pi i/3}$	$e^{2\pi i/3}$	i	i	$e^{2\pi i/3}$	$e^{2\pi i/3}$
H^2	$\widetilde{D_4}$	$\widetilde{E_8}$	$\widetilde{A_0}$	$\widetilde{E_7}$	$\widetilde{A_1}$	$\widetilde{E_6}$	$\widetilde{A_2}$

Then for any $l > 0$, the central fiber of

$$(\{(u,v) | \mathrm{arg} u \in [0, 2\pi\beta], |u| > 0\} \subset (\mathbb{C} - \{0\}) \times \mathbb{C}/(\mathbb{Z}l \oplus \mathbb{Z}\tau l))/\sim,$$

can be resolved (by Kodaira), where $(u,v) \sim (e^{2\pi i\beta}u, e^{-2\pi i\beta}v)$
The resolution is called an isotrivial ALG space. It can be compactified into a rational elliptic surface minus a singular fiber at infinity.

Gibbons-Hawking construction

Example (Gibbons-Hawking)

Let U be a subset of $\mathbb{R}^3$, $\mathbb{R}^2 \times S^1$, or $\mathbb{R} \times T^2$, g_U be the flat metric on U, and V be a harmonic function on U. Suppose that there is an S^1 fibration E on U such that the connection 1-form α satisfies

$$d\alpha = *_{g_U} dV,$$

then the metric

$$g_E = V g_U + V^{-1} \alpha^2$$

is a hyperKähler metric on E.

Gibbons-Hawking construction

Example

If $U = \mathbb{R}^3 \setminus \{0\} = (0, \infty) \times S^2$, $V = \frac{1}{4\pi r}$, and the S^1 fibration is the Hopf fibration $S^1 \to S^3 \to S^2$, then the Gibbons-Hawking metric is just the Euclidean metric on $\mathbb{R}^4 \setminus \{0\} = (0, \infty) \times S^3$.

Example (multi-Ooguri-Vafa)

When $U = (\mathbb{R}^2 \times \mathbb{S}^1) \setminus \{p_1, ..., p_\nu\}$ and V be a harmonic function on U such that $V \sim T - \frac{\nu}{2\pi} \log r$ for a constant T, and $V \sim \frac{1}{4\pi|x - p_i|}$ near p_i, then the Gibbons-Hawking metric is called the multi-Ooguri-Vafa metric.

Example (Eguchi-Hanson metric)

When $U = \mathbb{R}^3 \setminus \{p_1, p_2\}$ and $V = \frac{1}{4\pi|x-p_1|} + \frac{1}{4\pi|x-p_2|}$, then the Gibbons-Hawking metric is called the Eguchi-Hanson metric. It is ALE and is asymptotic to $\mathbb{R}^4/\mathbb{Z}_2$.

Gluing of ALG and ALG* gravitational instantons

If we allow the change of the complex structure, ALG* and non-isotrivial order 2 ALG gravitational instantons can also be used as building blocks of the gluing construction.

Theorem (C.-Viaclovsky-Zhang)

For any elliptic K3 surface, we can glue the Greene-Shapere-Vafa-Yau's semi-flat metric with order 2 ALG gravitational instantons near finite monodromy fibers, multi-Ooguri-Vafa metric near I_ν fibers, and ALG* gravitational instantons together with Gibbons-Hawking metrics near I_ν^* fibers to get a hyperKähler metric on the K3 surface.

Corollary (C.-Viaclovsky-Zhang)

The uniqueness parts of the order 2 ALG Torelli theorem and ALG* Torelli theorem hold.

Gluing of ALG* gravitational instantons

Example (C.-Viaclovsky-Zhang)

Near each I_ν^* fiber, the Greene-Shapere-Vafa-Yau's semi-flat metric looks like the $\mathbb{Z}_2$ quotient of a Gibbons-Hawking metric with $U = \mathbb{R}^2 \times \mathbb{S}^1$ and $V \sim T - \frac{\nu}{\pi} \log r$ for a constant T. Recall that an ALG* space looks like the $\mathbb{Z}_2$ quotient of a Gibbons Hawking metric with $U = \mathbb{R}^2 \times \mathbb{S}^1$ and $V \sim T + \frac{b}{\pi} \log r$ for a constant T. In our gluing construction, we choose V as a harmonic function on

$$U = (\mathbb{R}^2 \times \mathbb{S}^1) \setminus \{p_1, ...p_{b+\nu}, -p_1, ..., -p_{b+\nu}\}$$

such that $V \sim T + \frac{b}{\pi} \log r$ near 0, $V \sim T - \frac{\nu}{\pi} \log r$ near ∞, $V \sim \frac{1}{4\pi|x-p_i|}$ near p_i, and $V \sim \frac{1}{4\pi|x+p_i|}$ near $-p_i$.

Existence part of Torelli-type theorems

- Conjecturally, the Biquard-Boalch's study of Higgs bundles with irregular singularities may provide enough ALG and ALG* gravitational instantons. They only proved that the metric on the moduli space is hyperKähler and complete.
- Fredrickson-Mazzeo-Swoboda-Weiss proved that the moduli space of Higgs bundles with parabolic singularities is ALG.
- Conjecturally, the Cherkis-Kapustin's periodic monopole construction may provide enough ALG* gravitational instantons. The only known result was the thesis of Foscolo, who proved that the moduli space is non-empty and the metric on the moduli space is hyperKähler.
- In a work with N.Z.Li in prepration, I'm proving that the Biquard-Boalch metric is ALG or ALG*.

Thanks

Thank you for your attention!